

**Mathematics Curricula and
Formative Assessments:
Toward an Error-based Approach to
Formative Data Use in Mathematics**

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Mathematics Curricula and Formative Assessments

Mathematics curricular approaches—including textbooks, computer-aided programs, and instructional processes—incorporate diagnostic, formative, and interim assessments to provide teachers with information about student understanding. These assessments are intended to lead to instructional change which, in turn, should lead to increased academic achievement. However, many teachers and schools are not using assessment results appropriately due to lack of time, lack of understanding regarding how to interpret the results, and broader misunderstandings about the nature of formative assessments (Good, 2011). As a result, professional development, improved formative assessment questions, and embedding those assessments into a process that uses this information can provide teachers with assistance in incorporating results into their curriculum. With the goal of providing a context to think about an error-based approach to formative data use in algebra and pre-algebra classrooms, this literature review begins with a discussion of specific curricular approaches to algebra instruction, moves to a description of particular types of assessments within those curricular contexts, and, finally, describes a common error-based approach to formative assessments that could be used universally across different mathematics curricula. Woven throughout the review is an emphasis on the different understandings of what is meant by formative assessment, and the importance of matching that understanding with a process for information use in order to ensure that the information derived from the assessments is useful and used by teachers in the classroom in ways that improve student learning.

Curricular Approaches to Algebra Instruction

There are many different approaches to teaching mathematics and these different teaching philosophies are reflected in the many curricula that are currently used within schools.

Understanding the general approaches present in these curricula, their relative effectiveness in improving student achievement, and the reasons for their effectiveness is critical to considering the ways in which formative assessment approaches can be used within these curricular approaches to improve student outcomes.

Textbooks

Slavin, Lake, and Groff (2009) place mathematics curricula into three categories: textbook based, computer based, and instructional process programs. The textbook based curricula can be broken down further and described as innovative programs, basic textbooks, or traditional textbooks. Innovative programs focus on *problem solving*, *alternative solutions*, and *conceptual understanding* (Slavin et al., 2009). Commonly used innovative programs include:

- The University of Chicago School Mathematics Project
- Connected Mathematics
- Core-Plus Mathematics

Such innovative programs can be contrasted with basic textbook strategies such as Saxon Mathematics which focus on a step-by-step approach to teaching mathematics fundamentals focusing on algorithmic solutions and repetitive opportunities for problem solving. Traditional textbooks bridge the space between the more innovative programs and the back-to-basics approach represented by Saxon. Traditional textbook publishers such as McDougal Littell and Prentice Hall add a focus on problem solving and conceptual skills to their traditional instruction in basic skills; however, the approaches that these texts use do not focus as strongly on alternative solutions and conceptual understanding as do the more innovative programs. In summary, while these three approaches teach similar content, they place their instructional emphases in different areas, all of which could benefit from meaningful and appropriate formative assessment procedures.

Computer-Aided Programs

Another key curricular approach adopted by schools to teach mathematics is computer-aided programs (Slavin et al., 2009). Programs such as Cognitive Tutor are self-paced replacements for the more traditional curricula discussed previously and are generally found in middle and high schools more often than elementary schools. These programs are generally individualized to the students' needs and allow for the teacher to provide assistance while the computer program regulates the students' progress. In contrast, other computer-based programs such as Compass Learning typically provide approximately 10 to 15 minutes of enrichment each day.

A third approach to teaching mathematics is computer-managed curriculum. These programs use computers as a tool for teachers' course planning and implementation. Through analysis of computer-based assessments teachers are supposed to use their professional judgment to make decisions about their practice. Accelerated Mathematics uses this approach to assess students and provide information for planning and practice. Such a computer-managed curriculum can be used in conjunction with professional learning communities or more innovative teaching methods to enhance the independent effects of the supported curriculum.

Instructional Processes

Finally, mathematics curricula can be defined by their instructional processes (Slavin et al., 2009). In this case, the content and emphases of the curriculum are relatively uniform—generally defined by state and National Council of Teachers of Mathematics standards—while delivery approaches differ. Teachers incorporate aspects of cooperative learning, meta-cognitive strategy instruction, individualized learning, mastery learning, or comprehensive school reform

into their practice in a consistent way (Slavin et al., 2009). The following are several examples of these processes as well as some discussion of their potential strengths and weaknesses.

Peer-Assisted Learning Strategies and Curriculum-Based Measurement (CBM) provide teachers with guided group work activities for students to develop inquiry and teamwork skills. CBM uses traditional diagnostic assessments in the form of skills analyses and provides information to practitioners about student understanding and growth. While this approach is conceptually important, there is a substantial weakness. Because only a few test questions reflect each skill, concepts are typically under-represented on the diagnostic assessments. This problem of under-representation produces a limited amount of information and, in turn, reduces the reliability of the instruments (Ketterlin-Geller & Yovanoff, 2009). This loss of reliability makes using the results of the assessments somewhat problematic, since teachers must make decisions based on unreliable instruments.

IMPROVE, a meta-cognitive strategy instructional program, incorporates aspects of cooperative learning and mastery learning into instruction. In IMPROVE, students work together and ask questions aloud in order to cohesively find similarities and differences among problems, find the best strategy, and reflect as a group. This meta-cognitive approach is intended to allow students to develop confidence in their mathematical ability and provide students the opportunity to create their own understandings of the material which is intended to generate broader conceptual understandings and decrease reliance on algorithmic approaches to mathematical problems.

In individualized instructional approaches, students work by themselves and are teacher-facilitated when they need assistance. This allows students to work at their own pace and have their progress monitored by their teachers. Mastery learning strategies such as I CAN seek to

have all students at the same level of concept mastery to ensure that all students are accomplishing the same goals. Students are assessed formatively and summatively in order to evaluate the achievement of this goal, monitor progress, and not move on to new content until all students are proficient.

Finally, comprehensive school reform programs, including Talent Development Middle School Mathematics and Talent Development High School, target high-poverty schools. These programs direct time heavily on reading and mathematics; incorporate manipulatives, discussion, hands-on activities, and real-life connections; and provide professional development for instructors. These approaches are intended to increase student engagement with mathematical concepts and provide real world examples of how mathematics could have utility in the lived experiences of students. The Partnership for Access to Higher Mathematics is an example of a program that targets at-risk students ready to enter high school, focuses on constructivist strategies, and provides social work interventions.

Table 1: Mathematics Programs by Type

Program Type	Mathematics Program																			
	Accelerated Mathematics	Carnegie Learning and Cognitive Tutor	Jostens/Compass Learning	Connected Mathematics	Core Plus	Curriculum-Based Measurement	Expert Mathematician	I CAN	IMPROVE	McDougal Littell	Partnership for Access to Higher Mathematics	Peer-Assisted Learning Strategies	PLATO	Prentice-Hall	Saxon Mathematics	Student Teams-Achievement Divisions	Talent Development Middle School Mathematics Program	Transition Mathematics	UCSMP	
Innovative textbook				•	•															
Basic textbook															•				•	•
Traditional textbook										•				•						
Supplemental computer-aided program (CAI)			•																	
Core CAI		•					•	•					•							
Computer-managed learning system	•																			
Cooperative learning						•			•			•				•				
Meta-cognitive									•											
Individualized																				
Mastery																				
Comprehensive school reform											•						•			

Source: Table created from Slavin et al. (2009) by authors.

Curriculum/Instructional Impact

Slavin et al. (2009) summarized the results of 100 studies on the effect of middle and high school mathematics programs. Their analysis revealed a single key finding: instructional differences were more consequential than curricular differences on measures such as standardized tests and state assessments. Further,

[p]rograms found to be effective with any subgroup tend to be effective with all groups. This suggests that educational leaders could reduce achievement gaps by providing research-proven programs to schools serving many disadvantaged and minority students. (Slavin et al., p. 887)

The weighted mean effect size¹ found by Slavin et al. (2009) was +0.03 for mathematics curricula, +0.10 for computer-assisted instruction, and +0.18 for instructional process strategies. Furthermore, the What Works Clearinghouse² (2006; 2007; 2008; 2009a; 2009b; 2009c; 2009d; 2009e; 2010a; 2010b; 2010c; 2010d; 2011) estimated similar effects for multiple middle school and high school mathematics programs. Slavin et al. (2009) also found that textbook choice made no difference, while cooperative learning strategies had strong impact. To date, the What Works Clearinghouse has only tested two algebra programs at the high school level. It must be noted that although the What Works Clearinghouse found Carnegie Learning and Cognitive Tutor to have no discernable effects, using different criteria and methods for evaluation,

¹ Slavin's meta-analysis defined effect size with the following scale: *Strong evidence of effectiveness*: At least one randomized or RQE study and one additional study, with a weighted mean ES of at least +0.20 and a collective sample size across all studies of at least 500 students. *Moderate evidence of effectiveness*: Two studies or multiple smaller studies with a collective sample size of 500 students, with a mean ES of at least +0.20. *Limited evidence of effectiveness*: At least one qualifying study with an ES of at least +0.10. *Insufficient evidence of effectiveness*: One or more qualifying study with nonsignificant outcomes and a median ES of less than +0.10.

² The What Works Clearinghouse (WWC) is an initiative created by the U.S. Department of Education's Institute for Education Sciences (IES) to establish what constitutes scientific evidence of successful educational interventions and programs.

Slavin et al. (2009) concluded that Cognitive Tutor did, in fact, have at least limited evidence of effectiveness. Table 2 summarizes the results of the work of Slavin and his colleagues.

Table 2: Effects of Middle School and High School Mathematics Programs

Program	Middle School Effect	High School Effect
Accelerated Math	No discernible effect	N/A
Carnegie Learning and Cognitive Tutor	Potentially positive effects (Average: +15 percentile points)	No discernible effect
Core Plus	N/A	Potentially positive effects (Average: +15 percentile points, Range: -15 to +36 percentile points) Positive (but not statistically significant) effect on SAT math scores
Expert Mathematician	Potentially positive effects (Average: +14 percentile points, Range +14 percentile points)	N/A
I CAN	Positive effects (Average: +5 percentile points, Range: -7 to +16 percentile points)	N/A
PLATO	No discernible effects (Average: -1 percentile point)	N/A
Saxon Mathematics	Mixed effects (Average: +9 percentile points, Range: +6 to +16 percentile points)	No discernible effects
Transition Mathematics	Mixed effects (Average: +0 percentile points, Range: -14 to +19 percentile points)	N/A
University of Chicago School Mathematics Program	No discernible effect (Average: -6 percentile points)	N/A

Note: No discernible effect refers to studies that tried to measure effects but did not find any. N/A refers to studies that did not attempt to measure effects at this level.

Source: Table created from What Works Clearinghouse (2006, 2007, 2008, 2009a, 2009b, 2009c, 2009d, 2009e, 2010a, 2010b, 2010c, 2010d, 2011) by authors.

Given these findings about the limited impact of curricula on student outcomes, and the key finding that teacher approach was more important to programmatic impact, a key tool that teachers have at their disposal is the correct and effective use of formative assessments. However, it is important to understand exactly what these assessments are and how they can be used, as well as their potential for improving the teaching and learning of mathematics. In the following section we begin the work of defining the types of assessments that are available to teachers in the classroom, their possible uses, and the research base that points to why these assessments haven't yet solved the problem of poor achievement in mathematics.

Types of Assessments (Diagnostic, Formative, and Interim)

There are many types of assessments that schools and teachers have at their disposal when they design their curricular plan for addressing mathematics achievement. However, quite often the distinctions between the assessments, their intended use, and their actual use become blurred, occasionally due to a lack of understanding about terminology and the benefits and limitations that these assessments imply. This section describes types of assessments and the ways they can be used by classroom teachers to improve teaching and learning.

Diagnostic Tests: Response Analysis

Diagnostic tests assess prior knowledge and skills and come in two forms: response analyses and cognitive diagnostic assessments. Response analyses provide information on mastery and understanding and allow instructors to alter instruction to address students' misunderstandings. Skills analyses can inform instructors of areas of difficulty when creating review activities, while error analyses may provide information to help plan re-teaching activities (Ketterlin-Geller & Yovanoff, 2009). Quizzes on computational facts such as decimal and fraction conversion can provide insight into which skills each student has mastered or partially mastered and which skills should be reviewed as a class. However, skills analyses do not reveal why the student did not answer the question correctly; therefore, error analyses are necessary for further information (Ketterlin-Geller & Yovanoff). For example, test questions on fraction conversions reveal information about computational skills and strategies.

Diagnostic Tests: Cognitive Diagnostics

Cognitive diagnostic assessments target specific cognitive processes and are used to design remedial programs or additional assistance (Ketterlin-Geller & Yovanoff, 2009).

Ketterlin-Geller and Yovanoff offer a sample cognitive diagnostic matrix, where each response item is attached to one or more cognitive attribute. The matrix includes information on which test items the student answers correctly and incorrectly, providing information on possible patterns in cognitive gaps (see Table 3).

Table 3: Sample Cognitive Diagnostic Items and Classification Matrix for Division of Fractions

Cognitive Attributes	Items																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Conceptual understanding of fractions	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Ability to convert mixed number to improper fraction				x	x	x	x				x			x			x	x		x
Ability to multiply fractions							x	x	x	x	x	x	x	x	x	x	x	x	x	x
Conceptual understanding of relationship between multiplication and division											x	x	x	x	x	x	x	x	x	x
Ability to apply inverse and multiply algorithm															x	x	x	x	x	x
Zachary's Responses (C = correct, I = incorrect)	C	C	C	I	C	C	I	C	C	C	I	C	C	C	C	I	I	I	I	I

Summary classification for Zachary:

Attributes Mastered:

- Conceptual understanding of fractions
- Ability to multiply fractions
- Conceptual understanding of relationship between multiplication and division

Focus of Supplemental Instruction (attributes not mastered):

- Ability to convert mixed number to improper fraction
- Ability to apply inverse and multiply algorithm

Source: Adapted from Ketterlin-Geller & Yovanoff (2009).

These types of assessments can be critical to school and teacher planning since they can provide important information that will allow teachers to plan and sequence their curriculum (or group students) in ways that match the particular strengths and weaknesses for specific groups. This allows teachers (or groups of teachers) to customize their approach or prepare lessons that address particular issues that are likely to vary from group to group. In addition, with enough information over time, teachers can isolate issues and topics that are sources of problems year after year, thus allowing for robust planning and research to address these chronic problems of misunderstanding.

Formative Assessment

Diagnostic assessments can be used formatively, but are somewhat distinct from true formative assessments in that they should not be administered often during a school year to avoid test–retest improvement through item familiarity. The National Mathematics Advisory Panel (2008) defines formative assessment as “the ongoing monitoring of student learning to inform instruction...[and] is generally considered a hallmark of effective instruction in any discipline” (p. 46). Formative assessment, by nature, is intended for instructional improvement and not to measure achievement or readiness and should be thought of as a process and not as individual instruments (Good, 2011).

Formative assessments can be informal or formal. Formal formative assessments are prepared instruments while informal formative assessments are typically spontaneous questions asked in class to check for student understanding (Ginsburg, 2009; McIntosh, 1997). Both approaches can be useful but are useful in different ways. Formal formative assessments are what most people think about when the topic is raised. These can be student quizzes, district benchmarks, or assessments created for a specific purpose.

The prepared nature of these assessments is both a strength and a weakness. On the plus side, since these assessments are prepared for particular purposes, they can be directly and thoughtfully linked to particular learning or curriculum theories. In addition, the interpretations of the data gathered from these assessments can be fixed ahead of time. For example, students who get questions 1 and 2 incorrect by choosing choices a) and c) can be quickly identified as making the same error in reducing fractions. However, on the down side, the highly structured nature of formal formative assessments lacks the real-time spontaneity that can be found in informal assessments. This can be a problem because it may limit the amount of feedback from a

student. For example, a quiz on solving two-step algebraic equations may reveal procedural misunderstanding (such as subtracting a variable from both sides instead of adding it to both sides) or operational errors (making computational mistakes). However, informal formative assessments, such as discourse, allow a teacher to instantly ask the student questions when a misunderstanding or error is assessed. For example, a teacher could ask, “*Why* do we add this to both sides of the equation?” or “Can you explain why we did this step?” This additional information provides teachers with more nuanced information that can be used to understand why a procedural or operational error was made, not simply whether such an error was made.

While informal formative assessments can be prepared ahead of time, the interpretation of the responses and follow-up questions generally occur in the course of a dynamic classroom session. Thus, these informal assessments can occur many times in a course session, and can be tailored to the issues that come up on a day-to-day basis. However, the real-time interpretation relies very heavily on the mathematical knowledge and skills of the teacher to select appropriate questions, follow-up thoughtfully, diagnose quickly, and make meaningful modifications in that course session or in subsequent lessons.

This distinction between formal and informal assessment is useful from a functional perspective, yet it is less useful as a pedagogical categorization since it encompasses so many different forms of assessment. In 1976, Piaget provided a more useful framework for teachers. He categorized formative assessments into three groups based on their form: observation, test, and clinical interview (as cited by Ginsburg, 2009).

Observation-based formative assessments intend to reveal information on “natural behavior” (Ginsburg, 2009, p. 112). This could include a conversation between two children about which number is ‘larger.’ Natural behavior may reveal informal or casual language use or

everyday interactions between two students that may differ if the students were required to answer a question or solve a problem in front of a teacher or classroom. However, Ginsburg argues that observations are highly theoretical and can be difficult in large classrooms settings and, thus, may have limited utility for teachers trying to improve student performance.

Task or test forms of formative assessments are pre-determined questions or projects given to some or all students that assess accuracy and problem solving strategy and are analogous to the formal assessments described previously. These types of formative assessment instruments can come in the form of worksheets, pop quizzes, mathematics journals, discourse, and student demonstrations.

Worksheets and pop quizzes can contain a number of questions that (like the diagnostic tests described earlier) can assess cognition through error and skill analysis. Student mathematics journals and student discourse about problems are additional formative assessment tools that allow students to directly express areas of concern and confusion and feelings toward instructional strategies and are not test-based. In addition, class discussions can help identify gaps in student understanding by allowing students to volunteer to speak or allowing the teacher to choose specific students to answer questions. Student demonstrations allow students to solve and explain problems in front of the class. Through this form, teachers can gain insight into student computational skills as well as student conceptual understanding through the student generated explanations. These brief formative assessments can be useful and reliable sources of information to check for student understanding but require a great deal of expertise developed by the teachers to capitalize on the information (Phelan, Kang, Niemi, Vendlinski, & Choi, 2009). Additionally, instant forms of formative assessments, including the use of electronic clickers, index cards, and individual whiteboards (where teachers can ask questions and students can

answer by holding up whiteboards) allow teachers to instantly re-teach topics where conceptual or computational errors exist (Crumrine & Demers, 2007).

Because task or test forms of formative assessments may not capture cognitive processes, clinical interviews can be conducted (Piaget, 1976, as cited by Ginsburg, 2009). An adaptation of clinical interviews appropriate in the mathematics education setting would begin with an observation of the student performing a pre-chosen task. The interviewee proposes a hypothesis about the behavior, assigns new tasks, then asks a series of questions that prompt answers to how the student is behaving or thinking. The interview should be student-centered and questions should be constructed in real time (Piaget). Effective clinical interviews are based on strong theory, hypotheses, and evidence (Piaget). Although interviews can provide more insight into student thinking than observations or tests, they are dependent on human skill and may not be reliable (Ginsburg, 2009).

Intended Use of Formative Assessment Information

Formative assessments can reveal information about a student's performance, thinking, knowledge, learning potential, affect, and motivation (Ginsburg, 2009). These assessments, when part of a structured process, may lead to significant increases in academic achievement (Black & Wiliam, 2009; Davis & McGowen, 2007). Black and Wiliam (1998) found that the use of formative assessments had an effect size between 0.4 and 0.7 standard deviation units and that, across 250 studies, areas of increased achievement all had the use of formative assessments in common.³

Effective instructional change based on formative assessment results can have multiple effects. First, these assessments can benefit the current cohort of students through instructional

³ Effect sizes greater than 0.4 are considered moderate to strong.

improvement tailored to their specific needs. Second, these instructional improvements remain available for future cohorts if their formative assessments reveal similar conceptual misconceptions or computational errors (Davis & McGowen, 2007). Black and Wiliam (2009) argue that using formative assessments must be an ongoing, iterative process because there is always room for improving the formative assessments as a guide to alter instruction and curriculum.

Interim (Benchmark) Assessments

Goertz, Oláh, and Riggan's (2009) study on interim assessments provides further insight into the impact and issues involving potential formative assessment instruments. Interim assessments tend to fall between formative and summative assessment instruments and are sometimes referred to as benchmark assessments. These benchmark-type assessment instruments are becoming more prevalent at the district level, and are often required of schools by districts. With appropriate use, interim assessments may serve as the means by which teachers can improve their instruction, track the effectiveness of curriculum, evaluate instructional programs, and predict a student's performance level at the end of the course (Perie, Marion, & Gong, 2009). Interim assessments differ from formative and diagnostic assessments in that they are typically administered every few months and are useful in evaluating school- or district-wide programs (Popham, 2008; Shepard, 2009). In addition, since these benchmark assessments are currently in wide-use and have largely been created by teachers and curriculum coordinators at the districts, they do not require new test development or additional testing. This simplifies their use and allows them to be more tightly coupled with the content taught by the district.

Teachers' Use & Misuse of Formative Assessments

A key problem with the use of formative assessments occurs after the design and implementation phase. Formative assessments are often viewed as an object, rather than a process by which student achievement and understanding can be improved through the use of assessment information (Good, 2011). According to Good, the phrase *formative use of assessment information* is more appropriate than the simple term *formative assessment*, largely because it places the emphasis on the important aspect of the assessments—the use of the information vs. the instruments themselves. However, this move from assessment data to data use is often the most difficult to manage in the classroom. More specifically, once a diagnostic or formative assessment has been administered, teachers are often unsure how to interpret and act upon the data (Dixon & Haigh, 2009).

According to Heritage, Kim, Vendlinski, and Herman (2009), teachers find it more difficult to make instructional changes from assessment results than to perform other tasks including using student responses to assess student understanding. This difficulty can result in poor utilization of the information provided by these assessment instruments. As Poon and Lung (2009) observe, “[T]eachers do not understand their students’ learning process well, and hence their teaching skills and methodology do not match the needs of these students” (p. 58).

Goertz et al. (2009) also found that the type of instructional change that teachers generally utilized in response to formative assessment results was deciding what topics to re-teach, with very little deviation in approach or targeting of specific conceptual misunderstandings. This approach, while responding to data generated by formative assessments, often did not utilize the full range of potential information available to them from the assessments. Moreover, the limit at which teachers chose to respond with instructional change

varied from school to school and even from teacher to teacher (Goertz et al., 2009). For example, in one classroom a teacher may use a classroom success rate of 80% percent while another teacher may use 60% percent as the threshold for re-teaching, causing differences from teacher to teacher regarding what level requires instructional change. In Heritage et al.'s (2009) study they found that the interaction between teachers' pedagogical knowledge, knowledge of mathematical principles, and mathematical tasks produced the largest error variability in teachers' knowledge of appropriate formative assessment use. This suggests that teachers with the most knowledge of the mathematical principles and tasks represented by the assessment knew how best to use the formative assessment instruments to inform their instructional practices.

Finally, teachers were affected by various factors when deciding how to alter instruction. For example, teachers often considered their own knowledge of individual students, how students performed compared to classmates, and their own perceptions about what students found challenging when they made their instructional decisions (Goertz et al., 2009). In addition, teachers in Goertz et al.'s study were not surprised by the results of the interim assessments and "they mentioned that the interim assessments largely confirmed what they already knew about student learning in mathematics" (p. 5). However, some teachers did follow-up with individual students in order to alter future instruction.⁴ These findings support those in Slavin et al. (2009) and provide a potential mechanism for why so much of teachers' instructional success was related to teacher choices in their approach to teaching.

While these findings may appear obvious (it makes sense that teachers who understand mathematics best would use the formative assessments best and that teachers take into account

⁴ While this is a potentially useful finding, Goertz et al. (2009) did not specify the ways in which teachers followed up with students.

their knowledge of their students), there are important implications for introducing formative assessment practices into schools. First, in schools where teachers do not have strong understandings of mathematical principles or the assessments themselves, the mere introduction of formative assessments is less likely to produce positive changes in classroom pedagogy. In addition, there are implications for program design. First, the types of assessment instruments introduced should be consistent with the level of knowledge and pedagogical sophistication of the teachers. That is to say, those formative instruments that require less mathematical sophistication to use appropriately should be introduced where appropriate in order to scaffold teachers towards the appropriate use of the more complicated formative assessment tools described previously. Second, the more intimately involved in the design of the formative assessment instruments the teachers are the more likely they are to understand the purpose of those assessments and, thus, the more likely they are to use them more appropriately and effectively. Finally, the more input teachers can have in the creation of the formative assessment instruments, the more directly they can tailor them to reflect the local priorities and the knowledge of their students that they possess.

Professional Development

Ginsburg (2009) argues that a main challenge in mathematics education is providing professional development opportunities on assessment. Goertz et al. (2009) argue that teachers who assessed conceptual understanding were more likely to respond with instructional change and incorporate more varied instructional methods, such as using arrays for multiplication or relating the steps used in two-digit subtraction to the steps necessary to complete a three-digit subtraction problem. Given this observed relationship, fostering these types of behaviors could be a topic for professional development. “Professional development for teachers should focus as

well on teacher content knowledge, developing teachers' instructional repertoires, and capacity to assess students' mathematical learning" (Goertz et al., p. 9). Furthermore, teachers and principals in Volante, Drake, and Beckett's 2010 study reported that professional learning communities (PLCs) provided the opportunity to discuss with other practitioners samples of student work and allowed discussion on consistent measurement. Thus, PLCs could be a useful structure to provide these professional development opportunities and link the assessments to the instructional practices that will address the assessments' findings.

Integrating Assessment Results with Curriculum and Instructional Change

It is imperative that mathematics curricula are designed to incorporate results from a variety of assessments (Goertz et al., 2009). Ginsburg (2009) states that the foundation of formative assessments is its capability to provide information that teachers can use to make instructional decisions. Table 4 describes the fit between particular curricular approaches or programs and specific assessment types, suggesting and illustrating the point that all assessment types do not fit with all curricular approaches. Popham (2008) categorizes the possible changes that can occur from the intentional integration of formative assessments:

- teacher's instructional change (teacher adjusts instruction based on assessment results)
- students' learning tactic change (students use results to adjust their own procedures)
- classroom climate change (entire classroom expectations for learning are changed)
- school-wide change (through professional development or teacher learning communities, the school adopts a common formative assessment).

Formative assessment test questions are not always written in such a way that allow for analysis of mathematics procedural and conceptual understanding (Goertz et al., 2009). For example, multiple-choice tests often contain distractors (or wrong answers) that help assess

common errors (e.g., in a mathematics problem asking for the area of a circle, distractors may include answers that used an incorrect area formula, a computational error, or a calculator inputting error). However, individual distractors may contain multiple errors, making it difficult for teachers to assess where the student made the mistake. In addition, the pattern of correct and incorrect answers could be used to look for specific misunderstandings and at the same time increase the reliability of the assessments (Shepard, Flexer, Hiebert, Marion, Mayfield, & Weston, 2005; Yun, 2005).

As discussed previously, formative test results are often difficult to interpret—even Piaget believed that he could not interpret the results of a standardized test because of the method of administration (Ginsburg, 2009). As a result, teachers often interpret student errors differently, resulting in differences in teacher responses to results (Goertz et al., 2009). In Goertz et al.'s study, responses to a student's error varied from procedural to conceptual explanations. For example, with regard to a question requiring a student to add two fractions, some teachers diagnosed it as a procedural error in which the student failed to find the common denominator, while others diagnosed it as a conceptual error in which the student failed to understand that the denominator indicated how many parts were in the whole. These differences in interpretation are important because each of these explanations would require different pedagogical approaches to address them.

These findings suggest that it is important that the design of formative assessments clearly reflect their intended use such that the number and types of explanations for incorrect responses could be mitigated through the design of the assessment tool, or through additional inquiry intended to differentially diagnose the reasons for the incorrect response. Further, the literature suggests that professional communities could be created for teachers to discuss the

specific differences in interpretation and come to a consensus about how to address them. In addition, constructs, format, and any supplemental component should align with state or district standards, and instructional strategies should align with the curriculum’s approach (Goertz et al., 2009). The broader principle underlying Goertz et al.’s work is that assessments should be used for a single purpose and, thus, tests intended for formative use may require the use of other tests to allow for evaluative and predictive purposes, such as a summative unit test or project (Goertz et al.).

Table 4: Best-Fit Assessments for Mathematics Curricular Approaches and Programs

Assessment Types	Curricular Approaches			Specific Programs								
	Textbook	Computer-assisted instruction	Instructional process programs	Accelerated Math	Cognitive Tutor	Core Plus	Expert Mathematician	I CAN	PLATO	Saxon Math	Transition Mathematics	UCSMP
Diagnostic Assessments	•	•	•	•	•	•	•	•	•	•	•	•
Observation	•		•			•					•	•
Clinical Interview	•		•									
Quiz	•	•	•		•	•	•	•	•	•	•	•
Worksheets	•		•			•		•		•	•	•
Journals	•					•				•	•	•
Discourse	•					•				•	•	•
Student Demonstration	•	•			•	•		•	•		•	•
Interim Assessments	•	•	•	•	•	•	•	•	•	•	•	•

Source: Table created from Ginsburg (2009) and Slavin et al. (2009) by authors.

One example of how this connection between assessment and pedagogy using error analysis could be accomplished would be to start from a list of common mathematical errors, link those errors to types of formative assessments, describe how those errors would be identified within those assessments, and determine what corrective action could be taken by teachers in the classroom. The next section will outline this process.

Common Errors in Algebra

In the late 1970s, Hendrik Radatz issued a call for action models for teachers to integrate diagnostic teaching and findings from educational and social psychology, claiming that an “analysis of individual differences in the absence of a consideration of the content of mathematics instruction can seldom give the teacher practical help for individualizing instruction or providing therapy for difficulties in learning a specific task” (Radatz, 1979, p. 164). Societal and curricular differences make this connection difficult and, thus, instructors should consider other factors such as the teacher, the curriculum, the environment, and interactions. Given these multiple forces involved in the learning of mathematics, analyzing errors “in the learning of mathematics are the result of very complex processes. A sharp separation of the possible causes of a given error is often quite difficult because there is such a close interaction among causes” (Radatz, 1979, p. 164).

In order to simplify this set of complex causes, mathematical errors have been classified into five areas: language errors; difficulty with spatial information; deficient mastery of prerequisite skills, facts, and concepts; incorrect associations and rigidity of thinking; and incorrect application of rules and strategies (Radatz, 1979). Common mistakes and misconceptions in algebra can be rooted in the meaning of symbols (letters), the shift from numerical data or language representation to variables or parameters with functional rules or patterns, and the recognition and use of structure (Kieran, 1989).

Language Errors

Language errors can have multiple sources including gaps in knowledge for English Language Learners (ELL) and English as a Second Language (ESL) students, as well as gaps in academic language knowledge. This is particularly true for all students working on word

problems. Students may lack reading comprehension skills that are required to interpret the information needed to solve a problem. Students may also have difficulty understanding academic language required to solve a problem. Prompts, word banks, and fill-in-the-blank questions may be used to help students solve open-ended questions. For example, a prompt and fill-in-the-blank could be used when asking a student to distinguish similarities and differences between polygons: “Squares and rectangles both have ___ sides but are different because _____.” Word banks can be used when defining properties of angles. For example, the words “acute,” “obtuse,” “vertical,” “equal” and “not equal” can be included with other terms in a word bank to help students fill-in the following sentences:

An angle that is less than 90 degrees is _____. (acute)

An angle that is greater than 90 degrees is _____. (obtuse)

_____ angles are formed when two lines intersect and have _____ measurements.
(vertical, equal)

Spatial Information Errors

Difficulties in obtaining spatial information can also cause errors. A strong correlation was found between spatial ability and algebraic ability (Poon & Leung, 2009). When problems are represented using icons and visuals, mathematics assessments assume students can think spatially. For example, students may make errors on questions about Venn diagrams due to difficulties in understanding that lines represent boundaries and may ignore the lines. “Perceptual analysis and synthesis often make greater demands on the pupil than does the mathematical problem itself” (Radatz, 1979, p. 165). Without considering this lack of spatial ability as a possible cause of the incorrect responses, teachers may invest a lot of time and energy in presenting new materials that would not address the root cause of the problem.

Poor Prerequisite Skills/Flexibility/Procedural Errors

When a student does not possess the necessary prerequisite skills, facts, and concepts to solve a problem, he or she will not be able to solve the problem correctly. For example, if a student does not know how to combine like-terms, he or she may face difficulty solving multi-step equations involving combining like-terms. Difficulties due to incorrect associations or rigidity of thinking are also common areas of error in mathematics. “Inadequate flexibility in decoding and encoding new information often means that experience with similar problems will lead to habitual rigidity of thinking” (Radatz, 1979, p. 167). Further, students make procedural errors when they incorrectly apply mathematical rules and strategies. Rushed solutions and carelessness can also cause errors. Interviews revealed that errors in simplifying expressions were caused by carelessness and could be fixed with improved working habits (Poon & Leung, 2009). In addition, many students do not have linear problem solving skills. In fact, for many students, when reaching a point of difficulty in a problem, they go back and change their translation of the problem to avoid the difficulty (VanLehn, 1988, as cited by Sebrechts, Enright, Bennet, Martin, 1996).

Table 5 includes examples of common algebra errors, the type of error it is caused by, the best type of assessment and example that could be used to address the error and, in most cases, examples of possible next steps that can be taken after the assessment to address the identified problems.

Table 5: Common Algebra Errors and Best-Fit Assessment Examples

Concept	Fractions
Common Algebra Error	$\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$
Caused by	Misunderstanding of fraction rules (procedural)
Best-Fit Assessment(s)	Discourse Quiz or Test Items Student Journal Clinical Interview
Example Assessment(s)	<p>Discourse: Ask students, does $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$? Have students raise hand for “yes” or “no.”</p> <p>Open-Ended Quiz or Test Item:</p> <p>Does $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$? Explain why or why not.</p> <p>True or False Quiz or Test Item:</p> $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$ <p>A) True B) False</p> <p>Multiple-Choice Quiz or Test Item:</p> <p>The expression $\frac{a}{b+c}$ is always equal to:</p> <p>A) $\frac{a}{b} + \frac{a}{c}$</p> <p>B) $\frac{b+c}{a}$</p> <p>C) Both A and B. D) None of the above.</p> <p>Student Journal: Does $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$? In complete sentences, explain why or why not. (<i>Hint:</i> Think of an example of when the equation is not true.)</p>

	<p>Clinical Interview: Observe the student simplifying fractions with variables. If the hypothesis is that the student simplifies the fraction incorrectly such that $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$, then the interview could ask questions such as, “Why did you simplify this step like that?” or “What do you think would happen if you plugged in different numbers for these variables? Would it still be true?”</p>
Possible Instructional Next Step	<p>Use values for a, b, and c. For example, $a = 8$, $b = 2$, $c = 4$. Walk students through the incorrect expression using those values:</p> $\text{Does } \frac{8}{2+4} \text{ equal } \frac{8}{2} + \frac{8}{4}?$ <p>Lead students to notice that:</p> $\frac{a}{b+c} = \frac{8}{2+4} = \frac{8}{6} = \frac{4}{3}$ <p>However,</p> $\frac{a}{b} + \frac{a}{c} = \frac{8}{2} + \frac{8}{4} = 4 + 2 = 6$ <p>Teachers can also lead students to consider if the following is true:</p> $\frac{3}{5} = \frac{3}{2+3}$ <p>Teachers can then ask if:</p> $\frac{3}{2+3} = \frac{3}{2} + \frac{3}{3}$ <p>To help visual learners, teachers can also use fraction pieces to explain that the two expressions are not equal. For example, a piece equivalent to $\frac{8}{2}$ plus a piece equal to $\frac{8}{4}$ are not equal.</p>

Concept	Fractions
Common Algebra Error	<p>“Creative Cancelling” (Rossi, 2008, p. 555)</p> $\frac{x + y^2}{x^2} = \frac{1 + y^2}{x}$ <p>or</p> $\frac{x^2}{x + y^2} = \frac{x}{1 + y^2}$
Caused by	Misunderstanding of fraction rules (procedural)
Best-Fit Assessment(s)	Discourse or Student Journal Open-Ended Quiz or Test Item
Example Assessment(s)	<p>Discourse or Student Journal: Ask students, “Can you simplify $\frac{x + y^2}{x^2}$? Why or why not?”</p> <p>Open-Ended Quiz or Test Item: Can you simplify $\frac{x + y^2}{x^2}$? Why or why not?</p>
Possible Instructional Next Step	<p>Plug in values for x and y to illustrate how the expression can be simplified. For example, let $x = 3$ and $y = 4$.</p> $\text{Then } \frac{x + y^2}{x^2} = \frac{3 + (4)^2}{3^2} = \frac{19}{9}$ <p>Have students check that if you decompose the fraction you get the same answer:</p> $\frac{x + y^2}{x^2} = \frac{x}{x^2} + \frac{y^2}{x^2} = \frac{3}{3^2} + \frac{4^2}{3^2} = \frac{3}{9} + \frac{16}{9} = \frac{19}{9}$ <p>Have students verify that the “creative cancelling” does not yield the same answer:</p> $\frac{x + y^2}{x^2} = \frac{1 + y^2}{x} = \frac{1 + 4^2}{3} = \frac{17}{3}$ <p>Teachers can emphasize that fractions can only be cancelled when they are products.</p> <p>Teachers can also write multiple expressions on the board and have student volunteers solve the problems. Then, as a class, the teacher could guide the students in explaining why each step was made, what (if any) errors were made, why those errors are incorrect, and how to correctly proceed.</p>

Concept	Combining Like Terms						
Common Algebra Error	<p>Simplify</p> $-(6x + y) + x^2 + y$ $= -6x - y + x^2 + y$ $= -6x + x^2 - y$ $= -5x - y$ <p>(Poon & Leung, 2009, p. 53)</p>						
Caused by	“Weakness in algebraic manipulation skills and confusion of meaning of symbols and operations,” inventing strategies based off strategies for simpler problems, over-generalizing rules (Poon & Leung, 2009, p. 54)						
Best-Fit Assessment(s)	Open-Ended Quiz or Test Item Student Journal						
Example Assessment(s)	<p>Open-Ended Quiz or Test Item: Simplify: $-(6x + y) + x^2 + y$. Look out for errors combining similar variables of different degrees.</p> <p>Student Journal: Have students simplify an expression and justify each step. Using a table similar to the one shown below could help students organize data:</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>Step</th> <th>Explanation</th> </tr> </thead> <tbody> <tr> <td>$-(6x + y) + x^2 + y$</td> <td>Original Expression</td> </tr> <tr> <td>$-6x - y + x^2 + y$</td> <td>I distributed the negative sign to the variables inside the parentheses.</td> </tr> </tbody> </table>	Step	Explanation	$-(6x + y) + x^2 + y$	Original Expression	$-6x - y + x^2 + y$	I distributed the negative sign to the variables inside the parentheses.
Step	Explanation						
$-(6x + y) + x^2 + y$	Original Expression						
$-6x - y + x^2 + y$	I distributed the negative sign to the variables inside the parentheses.						

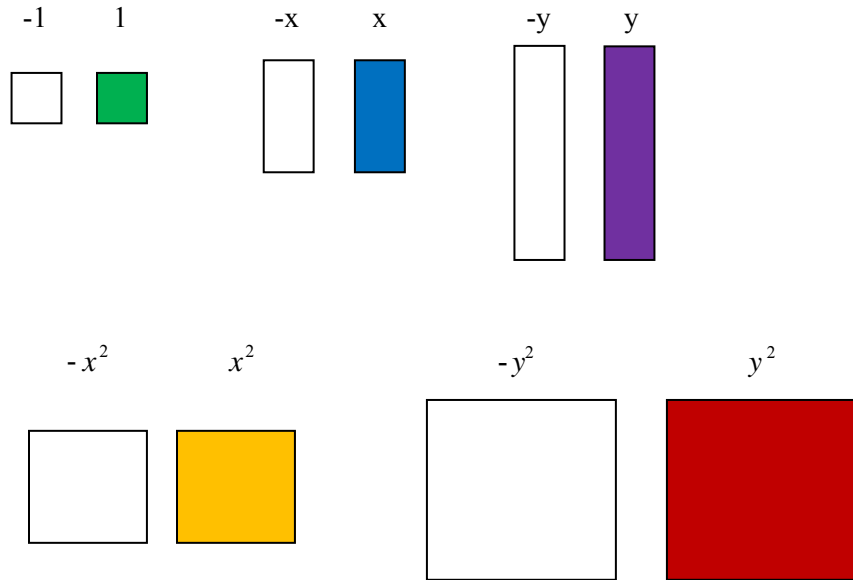
Possible
Instructional
Next Step

First have students simplify the expression. For example,

$$\begin{aligned} & -(6x + y) + x^2 + y \\ & = -6x - y + x^2 + y \end{aligned}$$

Once students simplify the expression, have students represent the expression using algebra tiles.

Use algebra tiles:



The colored tile represents the positive quantity and the white tile represents the negative quantity.

So in our example,

$$-6x - y + x^2 + y$$

This would represent six white “x” tiles, one white “y” tile, one colored “ x^2 ” tile, and one colored “y” tile. Have students combine the colored and white tiles that are the same size, which would be the colored “y” and white “y” tile. Students will then see that the only one colored “ x^2 ” tile and six white “x” tiles remain. Students will be able to see that the expression simplifies to: $= x^2 - 6x$

Teachers can use different colored markers or shapes to visually show that only “like terms” can be combined. For example, a triangle could be drawn around all x’s, circles around y’s, and squares around x^2 ’s. The x’s could be written in red, y’s in green, and x^2 ’s in blue. Students can then see that only similar colors can be added or subtracted.

Concept

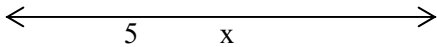
Simplifying Expressions

Common Algebra Error	$6x + 4 = 10x$ <i>or</i> $4x^2 + 2x = 6x^3$						
Caused by	Conceptual error						
Best-Fit Assessment(s)	Open-ended questions asking students to evaluate answers after simplifying (Poon & Leung, 2009) Student Journal						
Example Assessment(s)	<p>Open-Ended Quiz or Test Item: Simplify $4x^2 + 2x(x + 1)$. Check your answer by plugging in a value for x into the original expression and comparing it to the calculation from your final answer.</p> <p>Student Journal: Have students simplify an expression and justify each step. Using a table similar to the one shown below could help students organize data:</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>Step</th> <th>Explanation</th> </tr> </thead> <tbody> <tr> <td>$-(6x + y) + x^2 + y$</td> <td>Original Expression</td> </tr> <tr> <td>$-6x - y + x^2 + y$</td> <td>I distributed the negative sign to the variables inside the parentheses.</td> </tr> </tbody> </table>	Step	Explanation	$-(6x + y) + x^2 + y$	Original Expression	$-6x - y + x^2 + y$	I distributed the negative sign to the variables inside the parentheses.
Step	Explanation						
$-(6x + y) + x^2 + y$	Original Expression						
$-6x - y + x^2 + y$	I distributed the negative sign to the variables inside the parentheses.						
Possible Instructional Next Step	<p>Use algebra tiles (see above)</p> <p>Teachers can emphasize that a variable represents a value and each variable represents a different value. For example, a teacher could use the expression “$6a + 7c$” and say “6 apples and 7 carrots. Can we add them if they are different?”</p>						

Concept	Solving inequalities
Common Algebra Error	<p>Solve the inequality</p> $\frac{x+2}{5} + 1 < \frac{2x+3}{5}$ $= 5(x+2) + 1 + 5(2x+3)$ $= 5x + 10 + 1 + 10x + 15$ $= \frac{15x}{15} + \frac{31}{15}$ $= x + \frac{31}{15}$ <p>(Poon & Leung, 2009, p. 54)</p>
Caused by	Weakness in syntax error
Best-Fit Assessment(s)	Open-Ended Quiz or Test Item Student Demonstration
Example Assessment(s)	<p>Open-Ended Quiz or Test Item: Solve for x: $\frac{x+2}{5} + 1 < \frac{2x+3}{5}$</p> <p>Student Demonstration: Students solve the question on an individual whiteboard and after an allotted amount of time, all students raise their boards with the final answer circled.</p>
Possible Instructional Next Step	<p>Review the concept of having a common denominator (i.e., have 1 represent 5/5)</p> <p>Teachers can have students identify errors in the above problem, asking the class questions such as, “What error was made here? What should we do at this step? Can we reduce this? How do we reduce this? Can we combine these?”</p> <p>Teachers should solve the problem clearly and legibly in a line-by-line process so that students can see each step of the solution.</p>

Concept	Word problem (mistranslation)
Common Algebra Error	The length of Sarah's room is 10 feet. Sarah's room is 18 inches wider than the length of the room. What is the area of Sarah's room? Student's Answer: $(10)(18) = 180 \text{ ft}^2$
Caused by	Weak understanding of terms, low levels of linguistic knowledge to translate what is being asked (Sebrechts et al., 1996)
Best-Fit Assessment(s)	Multiple-Choice Quiz or Test Item Open-Ended Quiz or Test Item Involving Drawing Diagrams Clinical Interviews
Example Assessment(s)	Multiple-Choice Quiz or Test Item: The length of Sarah's room is 10 feet. Sarah's room is 18 inches wider than the length of the room. What is the area of Sarah's room? A) 180 ft^2 (student multiplied two numbers presented in word problem) B) 28 ft^2 (student added two numbers presented in word problem) C) 280 ft^2 (student thought the width was $10+18=28\text{ft.}$) D) 115 ft^2 (correct answer) Open-Ended Quiz or Test Item Involving Drawing Diagrams: The length of Sarah's room is 10 feet. Sarah's room is 18 inches wider than the length of the room. Draw and label a diagram then find the area of Sarah's room.
Possible Instructional Next Step	Because this is an error caused by misreading the question, teachers can re-teach by reminding students to read questions carefully. Teachers can utilize a number of different strategies to help students organize their information. For example, teachers can ask students to list the following when solving word problems: What do we know? What do we want to know? What equation(s) do I know that will help me solve this?

Concept	Carelessness or “slips”
Common Algebra Error	<p>Simplify</p> $2a + 5[(6a + 8)2 + 3a]$ $= 2a + 5[12a + 16 + 3a]$ $= 2a + 5[15a + 16]$ $= 2a + 5[13a]$ $= 2a + 155$ <p>(Greeno et al., 1985)</p>
Caused by	Carelessness (line 4)
Best-Fit Assessment(s)	Open-Ended Analysis Item “Error Analysis” Question Clinical Interview
Example Assessment(s)	<p>Open-Ended Item: Simplify: $2a + 5[(6a + 8)2 + 3a]$</p> <p>“Error Analysis” Question: Billy did not receive full credit on the problem he solved below. Explain what error(s) he made and where. Then, solve the problem correctly.</p> <p>Simplify</p> $2a + 5[(6a + 8)2 + 3a]$ <p>Billy’s Answer</p> $2a + 5[12a + 16 + 3a]$ $= 2a + 5[15a + 16]$ $= 2a + 5[13a]$ $= 2a + 155$ <p>Clinical Interview: If “careless” errors are assessed in a clinical interview, then the interviewee could ask questions to uncover this problem.</p>
Possible Instructional Next Step	Teachers can re-teach order of operations and use arrows to visually depict the distributive property. Teachers can also use algebra tiles, as explained in a previous example. Teachers can also write three to five example questions on the board and have student volunteers or chosen students solve them.

Concept	Word to Number Sentences
Common Algebra Error	Translate “five less than a number” into a number sentence: $5 - x$
Caused by	Reversal error
Best-Fit Assessment(s)	Multiple-Choice Discourse
Example Assessment(s)	Multiple Choice: Which of the following is a number sentence for the word sentence “five less than a number”? A) $x - 5$ (correct answer) B) $5 - x$ (student wrote number sentence left-to-right as it is written in the problem)
Possible Instructional Next Step	Use a number line to show the relationship in the expression. For example, “five less than a number,” first ask the student to plot a number, let’s say “x,” on a number line. Then have the student determine where to place the number 5 in relation to “x” (i.e., should it be placed below x (less than x) or above x (more than x))? Students will hopefully see that “five less than a number” translates to “ $x - 5$ ” 

Concept	Drawing Lines
Common Algebra Error	Spatial Sense
Caused by	Inability to visualize lines on a plane
Best-Fit Assessment(s)	Open-Ended Quiz or Test Item Discourse Clinical Interview

<p>Example Assessment(s)</p>	<p>Open-Ended Quiz or Test Item: James is walking up a hill that has a consistent slope. The bottom of the hill is at sea level and by the time he reaches the top of the hill, he is at 40 meters above sea level. He walked 40 meters total to the top of the hill. Draw a line on a coordinate plane to show his travel. Then, find the slope and equation of the line that best describes his travel.</p> <p>Clinical Interview: Ask the student a series of questions in which he or she is required to depict a line given information about slope and distance. Then ask the students questions such as, “What do you think is meant by ‘sea level?’” or “What is the difference between being 40 meters above sea level and walking 40 meters to the top of the hill?” These questions will help determine what error the student is making and investigate his or her understanding of the given information.</p>
<p>Possible Instructional Next Step</p>	<p>While going through word problems that do not include a diagram, lead and encourage students to draw diagrams to accompany each word problem. Have students act out the scenario to help understand the direction of travel and what each piece of information contributes to the word problem.</p>

<p>Concept</p>	<p>Rates</p>
<p>Common Algebra Error</p>	<p>Academic Language</p>
<p>Caused by</p>	<p>Misunderstanding of multiple-meaning words used in everyday life and in mathematics</p>
<p>Best-Fit Assessment(s)</p>	<p>Quiz or Test Item Clinical Interview</p>

<p>Example Assessment(s)</p>	<p>Quiz or Test Item: Example of multiple-meaning word confusion:</p> <p style="padding-left: 40px;">Juan and Kailan <i>walked</i> twenty miles for charity, and <i>in so doing</i> they raised \$200 for the Children’s Foundation. How much money did the charity <i>walk</i> raise per mile? (Scarcella, 2010)</p> <p>English Language Learner students may find the second use of the word walk confusing. Further, the phrase “in so doing” may not be familiar to ELL students. Omitting or replacing the word and phrases may reduce confusion:</p> <p style="padding-left: 40px;">Juan and Kailan walked twenty miles for charity, and raised \$200 for the Children’s Foundation. How much money did they raise per mile?</p> <p>Clinical Interview: Ask the student to solve a series of word problems to assess for word confusion. After the student reads a word problem, ask the student to rephrase it and restate what the question is asking. Prompt the student to answer questions, such as:</p> <p style="padding-left: 40px;">What do we know?</p> <p style="padding-left: 40px;">What do we want to know?</p> <p style="padding-left: 40px;">What equation(s) do we need to solve?</p>
<p>ossible Instructional Next Step</p>	<p>Teachers can avoid using multiple-meaning words when creating their own worksheets or tests or can rephrase word problems provided in curricula. In order to assist students in understanding word problems, teachers can ask students to list the following when solving word problems:</p> <p style="padding-left: 40px;">What do we know?</p> <p style="padding-left: 40px;">What do we want to know?</p> <p style="padding-left: 40px;">What equation(s) do we need to solve?</p>

The ability to use assessments in order to reduce common algebra errors may, in turn, increase understanding and build prerequisite skills that may lead to a stronger understanding of more advanced topics for both students and teachers alike. The use of open-ended quiz or test items allows teachers to see all of a student’s work rather than just an answer, such as in the case of multiple-choice questions. However, teachers who use quizzes or tests with multiple-choice questions provided in textbooks and other curricula could discuss in professional learning communities (PLC) what potential errors could have led a student to choose that multiple-choice

answer, whether it be procedural, conceptual, spatial, language, or random. From there, teachers may be able to see a pattern that arises among classes and share ideas on how to approach re-teaching. Teachers in a PLC setting could share and discuss common errors that have surfaced in their classrooms and what strategies have helped to address the errors.

Conclusion

Formative assessments are growing in importance; are critical in revealing student knowledge, motivation, and thinking; and have been part of various educational reforms in the past decade. Formative assessments can be both formal and informal. Some are more appropriate for particular types of curricula. Although information exists on the types and use of formative assessments, difficulties and misunderstandings persist regarding how to interpret the results of formative assessments and what instructional steps should be made following the interpretation. From formative assessments, teachers may be able to see which topics to re-teach yet may not have a clear understanding of how to alter instructional strategies to re-teach the topic. One way to address these issues is through a common-error approach which can most effectively be used within a collaborative teacher setting to come to a consensus about explanations for those errors and how to interpret the data generated. Understanding common errors in algebra could help teachers develop, interpret, and react to formative assessments. To help build a connection between common algebra errors, formative assessments, and instructional practice, this literature review aims to move toward a better understanding of how teachers can develop formative assessments to address common errors in algebra, how they can respond post-assessment to clarify misunderstandings, the importance of collaboration, and the possible role that professional learning communities can play in this overall process.

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